

# Weight Distributions of Hamming Codes

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**Abstract**—We derive a recursive formula determining the weight distribution of the  $[n = (q^m - 1)/(q - 1), n - m, 3]$  Hamming code  $H(m, q)$ , when  $(m, q - 1) = 1$ . Here  $q$  is a prime power. The proof is based on Moisio's idea of using Pless power moment identity together with exponential sum techniques.

**Index Terms**—Hamming code, weight distribution, Pless power moment identity, exponential sum.

## I. INTRODUCTION

The Hamming code is probably the first one that someone encounters when he is taking a beginning course in coding theory. The  $q$ -ary Hamming code  $H(m, q)$  is an  $[n = (q^m - 1)/(q - 1), n - m, 3]$  code which is a single-error-correcting perfect code. From now on,  $q$  will indicate a prime power unless otherwise stated. Also, we assume  $m > 1$ .

In [3], Moisio discovered a handful of new power moments of Kloosterman sums over  $\mathbb{F}_q$ , with  $q = 2^r$ . This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and frequencies of weights in the binary Zetterberg code of length  $q + 1$ , which were known by the work of Schoof and van der Vlugt in [7]. Some new moments of Kloosterman sums were also found over  $\mathbb{F}_q$ , with  $q = 3^r$  ([4],[8]).

In this correspondence, we adopt Moisio's idea of utilizing Pless power moment identity and exponential sum techniques and prove the following theorem giving the weight distribution of  $H(m, q)$ , for  $(m, q - 1) = 1$ .

**Theorem 1:** Let  $\{C_h\}_{h=0}^n$  ( $n = (q^m - 1)/(q - 1)$ ) denote the weight distribution of the  $q$ -ary Hamming code  $H(m, q)$ , with  $(m, q - 1) = 1$ . Then, for  $h$  with  $1 \leq h \leq n$ ,

$$h!C_h = (-1)^h q^{m(h-1)} (q^m - 1) + \sum_{i=0}^{h-1} (-1)^{h+i-1} C_i \sum_{t=i}^h t! S(h, t) q^{h-t} (q - 1)^{t-i} \binom{n-i}{n-t},$$

where  $S(h, t)$  denotes the Stirling number of the second kind defined by

$$S(h, t) = \frac{1}{t!} \sum_{j=0}^t (-1)^{t-j} \binom{t}{j} j^h. \quad (1)$$

$C_0 = 1$ , and it is easy to check that  $C_1 = C_2 = 0$ , as it should be. A few next values of  $C_h$ 's were obtained, with the help of *Mathematica*, from the above formula.

**Corollary 2:** Let  $\{C_h\}_{h=0}^n$  ( $n = (q^m - 1)/(q - 1)$ ) denote the weight distribution of the  $q$ -ary Hamming code  $H(m, q)$ ,

with  $(m, q - 1) = 1$ . Then

$$\begin{aligned} C_3 &= \frac{1}{3!} (q^m - 1) (-q + q^m), \\ C_4 &= \frac{1}{4!} (q^m - 1) (-6q + 5q^2 + 6q^m + q^{2m} - 6q^{1+m}), \\ C_5 &= \frac{1}{5!} (q^m - 1) (-36q + 54q^2 - 26q^3 + 36q^m + 6q^{2m} \\ &\quad + q^{3m} - 60q^{1+m} + 35q^{2+m} - 10q^{1+2m}), \\ C_6 &= \frac{1}{6!} (q^m - 1) (-240q + 500q^2 - 450q^3 + 154q^4 \\ &\quad + 240q^m + 20q^{2m} + 10q^{3m} + q^{4m} - 520q^{1+m} \\ &\quad + 85q^{2(1+m)} + 550q^{2+m} - 225q^{3+m} - 110q^{1+2m} \\ &\quad - 15q^{1+3m}), \\ C_7 &= \frac{1}{7!} (q^m - 1) (-1800q + 4710q^2 - 6035q^3 + 3940q^4 \\ &\quad - 1044q^5 + 1800q^m - 90q^{2m} + 85q^{3m} + 15q^{4m} \\ &\quad + q^{5m} - 4620q^{1+m} + 1505q^{2(1+m)} + 6755q^{2+m} \\ &\quad - 5215q^{3+m} + 1624q^{4+m} - 805q^{1+2m} - 735q^{3+2m} \\ &\quad - 245q^{1+3m} + 175q^{2+3m} - 21q^{1+4m}), \\ C_8 &= \frac{1}{8!} (q^m - 1) (-15120q + 47124q^2 - 77196q^3 + 72779q^4 \\ &\quad - 37240q^5 + 8028q^6 + 15120q^m - 3276q^{2m} + 840q^{3m} \\ &\quad + 175q^{4m} + 21q^{5m} + q^{6m} - 43848q^{1+m} \\ &\quad + 17934q^{2(1+m)} - 1960q^{3(1+m)} + 79632q^{2+m} \\ &\quad + 6769q^{2(2+m)} - 87808q^{3+m} + 52661q^{4+m} \\ &\quad - 13132q^{5+m} - 3276q^{1+2m} - 19236q^{3+2m} \\ &\quad - 3080q^{1+3m} + 4270q^{2+3m} - 476q^{1+4m} + 322q^{2+4m} \\ &\quad - 28q^{1+5m}), \\ C_9 &= \frac{1}{9!} (q^m - 1) (-141120q + 507024q^2 - 1002736q^3 \\ &\quad + 1221444q^4 - 910644q^5 + 382088q^6 - 69264q^7 \\ &\quad + 141120q^m - 57456q^{2m} + 10864q^{3m} + 1960q^{4m} \\ &\quad + 322q^{5m} + 28q^{6m} + q^{7m} - 449568q^{1+m} \\ &\quad + 165396q^{2(1+m)} - 67116q^{3(1+m)} + 957936q^{2+m} \\ &\quad + 246624q^{2(2+m)} - 1349404q^{3+m} + 1175874q^{4+m} \\ &\quad - 571116q^{5+m} + 118124q^{6+m} + 33936q^{1+2m} \\ &\quad - 332584q^{3+2m} - 67284q^{5+2m} - 39396q^{1+3m} \\ &\quad + 74844q^{2+3m} + 22449q^{4+3m} - 7812q^{1+4m} \\ &\quad + 10332q^{2+4m} - 4536q^{3+4m} - 840q^{1+5m} \\ &\quad + 546q^{2+5m} - 36q^{1+6m}), \end{aligned}$$

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and

$$\begin{aligned}
C_{10} = & \frac{1}{10!}(q^m - 1)(-1451520q + 5880384q^2 - 13550832q^3 \\
& + 20090832q^4 - 19485852q^5 + 11984244q^6 \\
& - 4251240q^7 + 663696q^8 + 1451520q^m - 893376q^{2m} \\
& + 174384q^{3m} + 21504q^{4m} + 4536q^{5m} + 546q^{6m} \\
& + 36q^{7m} + q^{8m} - 4987008q^{1+m} + 857520q^{2(1+m)} \\
& - 1569540q^{3(1+m)} + 63273q^{4(1+m)} + 12035088q^{2+m} \\
& + 5797770q^{2(2+m)} - 20393616q^{3+m} + 723680q^{2(3+m)} \\
& + 23050848q^{4+m} - 16423398q^{5+m} + 6661236q^{6+m} \\
& - 1172700q^{7+m} + 1341360q^{1+2m} - 4686480q^{3+2m} \\
& - 3264780q^{5+2m} - 576240q^{1+3m} + 1233960q^{2+3m} \\
& + 1030260q^{4+3m} - 269325q^{5+3m} - 117012q^{1+4m} \\
& + 227808q^{2+4m} - 196392q^{3+4m} - 17430q^{1+5m} \\
& + 22260q^{2+5m} - 9450q^{3+5m} - 1380q^{1+6m} \\
& + 870q^{2+6m} - 45q^{1+7m}).
\end{aligned}$$

The Hamming code was discovered by Hamming in late 1940's. So it is surprising that there are no such recursive formulas determining the weight distributions of the Hamming codes in the nonbinary cases. In the binary case, we have the following well known formula which follows from elementary combinatorial reasoning ([2, p. 129]).

*Theorem 3:* Let  $\{C_h\}_{h=0}^n$  ( $n = (2^m - 1)$ ) denote the weight distribution of the binary Hamming code  $H(m, 2)$ . Then the weight distribution satisfies the following recurrence relation:

$$\begin{aligned}
C_0 &= 1, \quad C_1 = 0, \\
(i+1)C_{i+1} + C_i + (n-i+1)C_{i-1} &= \binom{n}{i} \quad (i \geq 1).
\end{aligned}$$

It is known [6] that, when  $(m, q-1) = 1$ ,  $H(m, q)$  is a cyclic code.

*Theorem 4:* Let  $n = (q^m - 1)/(q - 1)$ , where  $(m, q-1) = 1$ . Let  $\gamma$  be a primitive element of  $\mathbb{F}_{q^m}$ . Then the cyclic code of length  $n$  with the defining zero  $\gamma^{q-1}$  is equivalent to the  $q$ -ary Hamming code  $H(m, q)$ .

In our discussion below, we will assume that  $(m, q-1) = 1$ , so that  $H(m, q)$  is a cyclic code with the defining zero  $\gamma^{q-1}$ , where  $\gamma$  is a primitive element of  $\mathbb{F}_{q^m}$ .

## II. PRELIMINARIES

Let  $q = p^r$  be a prime power. Then we will use the following notations throughout this correspondence.

$$tr(x) = x + x^p + \cdots + x^{p^{r-1}}$$

the trace function  $\mathbb{F}_q \rightarrow \mathbb{F}_p$ ,

$$Tr(x) = x + x^q + \cdots + x^{q^{m-1}}$$

the trace function  $\mathbb{F}_{q^m} \rightarrow \mathbb{F}_q$ ,

$$\lambda(x) = e^{\frac{2\pi i}{p} tr(x)}$$

the canonical additive character of  $\mathbb{F}_q$ ,

$$\lambda_m(x) = \lambda(Tr(x))$$

the canonical additive character of  $\mathbb{F}_{q^m}$ .

The following lemma is well known.

*Lemma 5:* For any  $\alpha \in \mathbb{F}_q$ ,

$$\sum_{x \in \mathbb{F}_q} \lambda(\alpha x) = \begin{cases} q, & \alpha = 0, \\ 0, & \alpha \neq 0. \end{cases}$$

For a positive integer  $s$ , the multiple Kloosterman sum  $K_s(\alpha)$  ( $\alpha \in \mathbb{F}_q^*$ ), is defined by

$$K_s(\alpha) = \sum_{x_1, \dots, x_s \in \mathbb{F}_q^*} \lambda(x_1 + \cdots + x_s + \alpha x_1^{-1} \cdots x_s^{-1}).$$

The following result follows immediately from Lemma 5.

*Lemma 6:* For an integer  $s > 1$ ,

$$\sum_{\alpha \in \mathbb{F}_q^*} K_{s-1}(\alpha) = (-1)^s.$$

*Proof:*  $\sum_{\alpha \in \mathbb{F}_q^*} K_{s-1}(\alpha) = (\sum_{x \in \mathbb{F}_q^*} \lambda(x))^s$ . ■

The following lemma is immediate.

*Lemma 7:* Let  $(m, q-1) = 1$ . Then the following map is a bijection.

$$\alpha \mapsto \alpha^m : \mathbb{F}_q^* \rightarrow \mathbb{F}_q^*.$$

*Theorem 8 (Thm. 3 of [5]):* For any  $\alpha \in \mathbb{F}_{q^m}^*$ ,

$$\sum_{x \in \mathbb{F}_{q^m}^*} \lambda_m(\alpha x^{q-1}) = (-1)^{m-1} (q-1) K_{m-1}(N(\alpha)),$$

where  $N$  denotes the norm map  $N : \mathbb{F}_{q^m}^* \rightarrow \mathbb{F}_q^*$ , defined by  $N(\alpha) = \alpha^n$ , with  $n = (q^m - 1)/(q - 1)$ .

The following theorem is due to Delsarte ([2, P. 208]).

*Theorem 9 (Delsarte):* Let  $B$  be a linear code of length  $n$  over  $\mathbb{F}_{q^m}$ . Then

$$(B|_{\mathbb{F}_q})^\perp = Tr(B^\perp).$$

The following is a special case of the result stated in [1, Thm. 4.2], although only the binary case is mentioned there. In fact, using Theorem 9 above, this can be proved in exactly the same manner as described immediately after the proof of Theorem 4.2 in [1].

*Theorem 10:* The dual  $H(m, q)^\perp$  of  $H(m, q)$  is given by

$$\begin{aligned}
H(m, q)^\perp &= \{c(a) = (Tr(a), Tr(a\gamma^{(q-1)}), \dots, Tr(a\gamma^{(n-1)(q-1)})) \\
&\quad | a \in \mathbb{F}_{q^m}\}.
\end{aligned}$$

*Lemma 11:* The map  $a \mapsto c(a) : \mathbb{F}_{q^m} \rightarrow H(m, q)^\perp$  is an isomorphism of  $\mathbb{F}_q$ -vector spaces.

*Proof:* The map is  $\mathbb{F}_q$ -linear, surjective and  $\dim_{\mathbb{F}_q} \mathbb{F}_{q^m} = \dim_{\mathbb{F}_q} H(m, q)^\perp$ . ■

Our recursive formula in Theorem 1 will be a consequence of the application of Pless power moment identity ([6]), which is equivalent to MacWilliams identity.

*Theorem 12 (Pless power moment identity):* Let  $B$  be an  $q$ -ary  $[n, k]$  code, and let  $B_i$  (resp.  $B_i^\perp$ ) denote the number of codewords of weight  $i$  in  $B$  (resp. in  $B^\perp$ ). Then, for  $h = 0, 1, 2, \dots$ ,

$$\sum_{i=0}^n i^h B_i = \sum_{i=0}^{\min\{n, h\}} (-1)^i B_i^\perp \sum_{t=i}^h t! S(h, t) q^{k-t} (q-1)^{t-i} \binom{n-i}{n-t}, \quad (2)$$

where  $S(h, t)$  denotes the Stirling number of the second kind defined by (1).

### III. PROOF OF THEOREM 1

Let  $h$  be an integer with  $1 \leq h \leq n$ . Observe that the weight of the codeword  $c(a)$  in Theorem 10 can be expressed as

$$\begin{aligned}
 w(c(a)) &= \sum_{i=0}^{n-1} (1 - q^{-1} \sum_{\alpha \in \mathbb{F}_q} \lambda(\alpha \text{Tr}(a\gamma^{i(q-1)}))) \\
 &\quad \text{(by Lemma 5)} \\
 &= n - q^{-1} \sum_{\alpha \in \mathbb{F}_q} \sum_{i=0}^{n-1} \lambda_m(\alpha a \gamma^{i(q-1)}) \\
 &= n - q^{-1} (q-1)^{-1} \sum_{\alpha \in \mathbb{F}_q} \sum_{x \in \mathbb{F}_{q^m}^*} \lambda_m(\alpha a x^{q-1}) \\
 &= n - q^{-1} (q-1)^{-1} (q^m - 1) - q^{-1} (q-1)^{-1} \\
 &\quad \times \sum_{\alpha \in \mathbb{F}_q^*} \sum_{x \in \mathbb{F}_{q^m}^*} \lambda_m(\alpha a x^{q-1}) \\
 &= n - q^{-1} (q-1)^{-1} (q^m - 1) + (-1)^m q^{-1} \\
 &\quad \times \sum_{\alpha \in \mathbb{F}_q^*} K_{m-1}(\alpha^m N(a)) \quad \text{(by Theorem 8)} \\
 &= n - q^{-1} (q-1)^{-1} (q^m - 1) + (-1)^m q^{-1} \\
 &\quad \times \sum_{\alpha \in \mathbb{F}_q^*} K_{m-1}(\alpha N(a)). \quad \text{(by Lemma 7)}
 \end{aligned} \tag{3}$$

We now apply Pless power moment identity in Theorem 12 with  $B = H(m, q)^\perp$ . On one hand, the LHS of (2) is

$$\begin{aligned}
 \sum_{a \in \mathbb{F}_{q^m}^*} w(c(a))^h &\quad \text{(by Lemma 11)} \\
 &= \sum_{a \in \mathbb{F}_{q^m}^*} (n - q^{-1} (q-1)^{-1} (q^m - 1) + (-1)^m q^{-1} \\
 &\quad \times \sum_{\alpha \in \mathbb{F}_q^*} K_{m-1}(\alpha N(a)))^h \quad \text{(by (3))} \\
 &= \frac{q^m - 1}{q - 1} \sum_{a \in \mathbb{F}_q^*} (n - q^{-1} (q-1)^{-1} (q^m - 1) + (-1)^m q^{-1} \\
 &\quad \times \sum_{\alpha \in \mathbb{F}_q^*} K_{m-1}(\alpha a))^h \\
 &= (q^m - 1) (n - q^{-1} (q-1)^{-1} (q^m - 1) + (-1)^m q^{-1} \\
 &\quad \times \sum_{\alpha \in \mathbb{F}_q^*} K_{m-1}(\alpha))^h \\
 &= (q^m - 1) (n - q^{-1} (q-1)^{-1} (q^m - 1) + q^{-1})^h \\
 &\quad \text{(by Lemma 6)} \\
 &= q^{(m-1)h} (q^m - 1) \quad \text{(as } n = \frac{q^m - 1}{q - 1} \text{)}.
 \end{aligned}$$

On the other hand, by separating the term corresponding to  $h$  and noting  $S(h, h) = 1$ , the RHS of (2) is

$$\begin{aligned}
 &(-1)^h C_h h! q^{m-h} \\
 &+ \sum_{i=0}^{h-1} (-1)^i C_i \sum_{t=i}^h t! S(h, t) q^{m-t} (q-1)^{t-i} \binom{n-i}{n-t}.
 \end{aligned}$$

So

$$\begin{aligned}
 &q^{(m-1)h} (q^m - 1) \\
 &= (-1)^h C_h h! q^{m-h} \\
 &\quad + \sum_{i=0}^{h-1} (-1)^i C_i \sum_{t=i}^h t! S(h, t) q^{m-t} (q-1)^{t-i} \binom{n-i}{n-t}.
 \end{aligned} \tag{4}$$

Multiplying both sides of (4) by  $(-1)^h q^{h-m}$ , we get the desired result.  $\blacksquare$

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